J. Fluid Mech. (1978), vol. 88, part 2, pp. 209–222
Printed in Great Britain

Focusing of weak shock waves at an arête

By M. S. CRAMER

Sibley School of Mechanical and Aerospace Engineering,
Cornell University, Ithaca, New York 14853†

AND A. R. SEEBASS

Department of Aerospace and Mechanical Engineering,
University of Arizona, Tucson, Arizona 85721

(Received 28 June 1976 and in revised form 13 February 1978)

The focusing of very weak and slightly concave symmetrical shock waves is examined. The equation that describes this focusing is derived and the resulting similitude discussed. The initial conditions come from a formal matching of this nonlinear description with the linear solution. The maximum value of the pressure coefficient is shown to be proportional to the two-thirds power of both the initial strength of the wave front and a parameter characterizing its rate of convergence.

1. Introduction

There are many sources of weak shock waves; they arise naturally and through the activities of man. Examples include the commonly experienced phenomena of thunder and the sonic bang generated by supersonic aircraft. In this paper we use the terms shock wave and wave front interchangeably to refer to a surface of discontinuity in the pressure and velocity fields in the fluid. It frequently happens that such wave fronts become curved. This curvature may be the result of inhomogeneities in the medium, reflexion from curved surfaces or unsteady boundary conditions. Wave fronts which are concave in the direction of propagation exhibit different kinds of behaviour depending upon the strength of the wave front and the rate of focusing. When the focusing is weak relative to the magnitude of the pressure rise across the wave front, the wave front will straighten and no focusing will occur. When the strength of the wave front is sufficiently small, the wave front will focus along a caustic surface and at a cusp in this surface, called an arête, if it occurs. A perfect focus occurs when a finite portion of the wave front converges to a single point.

The focusing process is characterized by large pressure amplification and a nonlinear interaction between the shock and the flow behind it. Despite considerable analytical, numerical and experimental work, many important questions remain unanswered. Analytical studies are hampered by the fact that available theories, such as the shock dynamic theory of Whitham (1957) and the theory of geometrical acoustics, are inapplicable at a focus. The first fails because it does not account for the interaction between the shock and the flow behind it; the second fails because it is a linear theory and predicts infinite pressures at focal points. Important theoretical studies of the

† Present address: Department of Aerospace and Mechanical Engineering, University of Arizona, Tucson, Arizona 85721.

behaviour of weak shock waves at a caustic include those of Guiraud (1965) and Hayes (1968). For the case of a smooth caustic they gave the similitude that relates the amplification of the wave front to its initial strength and geometry. An important contribution to our understanding of the behaviour of focusing wave fronts comes from the experimental investigations of Sturtevant & Kulkarny (1974, 1976). Using shadowgraph techniques and pressure measurements they studied the focusing of curved shocks for a wide range of geometries and strengths and delineated the complicated wave patterns and pressure histories which occur.

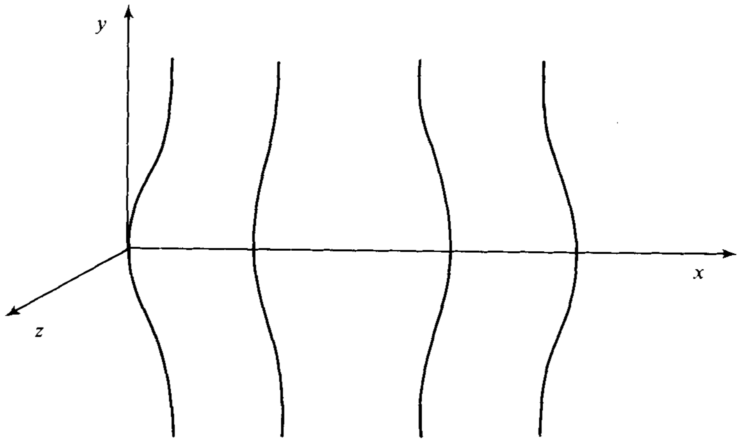
Because the caustics associated with smooth wave fronts are generally cusped, the arête is one of the more frequently observed foci. In this paper the focusing of very weak shocks at an arête is examined. We consider nearly straight symmetrical wave fronts and use the method of matched asymptotic expansions to determine the initial-value problem and related similitude that govern the flow in the vicinity of the arête. The dependence of the maximum pressure coefficient on the initial strength and shape of the wave front is discussed; the main result is that the pressure levels at the arête are proportional to the two-thirds power of both the initial strength of the wave front and a parameter characterizing the rate at which the wave-front converges.

2. Physical problem

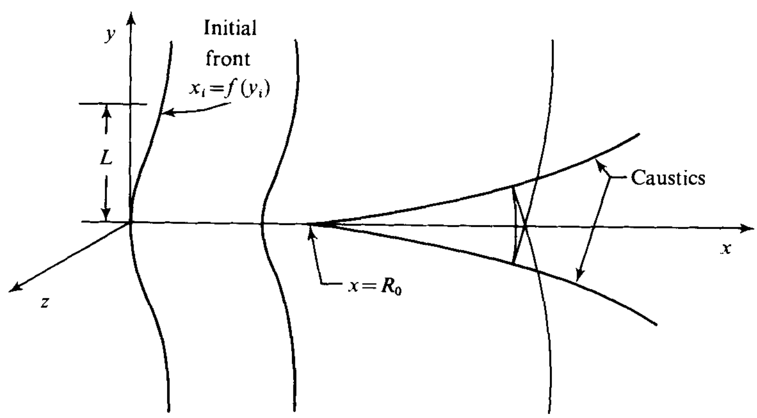
In this section we give a qualitative account of the physical effects that govern the propagation of curved wave fronts. One effect is that a wave front always propagates normal to itself and therefore has a tendency to converge. Another is that the speed of propagation of the wave front increases monotonically with its strength. The latter effect will tend to straighten converging shocks. The behaviour that results depends on which effect dominates.

The behaviour of shock waves which are relatively strong was discussed by Whitham (1957, 1959, 1974) using his theory of shock dynamics. To understand the behaviour of such shock waves, consider the propagation of a concave symmetrical shock into a homogeneous medium as sketched in figure 1 (*a*). The shock's strength is taken to be a maximum in the plane of symmetry and, because the amplification is greatest in this plane, the maximum strength will remain there. However, the shock speed will also have its maximum in the plane of symmetry and the resultant variation in propagation speed will straighten the shock. This behaviour is expected even for shocks with pressure coefficients much less than one, provided that the focusing is sufficiently slow.

When the strength of the shock is sufficiently small its speed of propagation is approximately the sound speed of the undisturbed medium. In many respects the flow will resemble that predicted by geometrical acoustics, and we first discuss the behaviour of such weak shocks from this viewpoint. In this approximation the propagation speed is taken to be the sound speed of the undisturbed medium and, if we take this to be homogeneous, the trajectories of points on the shock, called rays, will be straight. Adjacent rays originating on concave portions of the wave front will intersect, and the locus of these intersections will form a surface in space, called a caustic. When the wave front has a minimum radius of curvature the caustic will be cusped and the wave front will emerge from the cusp in a crossed configuration, as sketched in figure 1(*b*). Experimental evidence of weak shocks which focus and cross is found in the sonic-bang measurements made by Wanner *et al.* (1972) and in the laboratory



(*a*)



(*b*)

FIGURE 1. Focusing of (*a*) a moderate strength shock and (*b*) an acoustic discontinuity; $f''(L) \equiv 0$, $R_0 = 1/f''(0)$.

investigations by Beasley, Brooks & Barger (1969), Cornet (1972) and Sturtevant & Kulkarny (1974, 1976).

Geometrical acoustics predicts that when the wave front reaches the caustic surface its pressure jump becomes infinite. At the cusp in the caustic this singularity is even stronger because portions from either side of the plane of symmetry focus there simultaneously. Of course, these singularities are never observed, and they merely indicate a local failure of the geometrical-acoustics approximation. Sturtevant & Kulkarny have discussed the nonlinear effects which limit the shock strength to finite values in the focal region. They showed that as the shock approaches a cusp in the caustic it is immediately followed by a sharp expansion. This is due to the amplification of the shock relative to the rest of the flow. When the pressure gradient behind the shock becomes sufficiently great the weakening effect of the expansion as it overtakes the shock becomes noticeable, even if the shock is still weak. Because of the shock's interaction with the expansion wave, the strength of the shock is limited to finite values.

3. Mathematical formulation

We consider the wave front shown in figure 1 (*b*) and take the co-ordinate system to have its origin at the point of minimum radius of curvature of the initial wave front. The $+x$ axis is in the direction of propagation and the y axis is tangential to the initial wave front at $y = 0$. The equation of the initial wave front is taken to be $x_i = f(y_i)$, where x_i and y_i are the co-ordinates of the initial shock and f is symmetrical about $y = 0$. We are considering the propagation of very weak shocks in an inviscid perfect gas with no heat conduction. In the case considered here the shock strength is always small. The results of Hayes (1957) may be used to argue that for the spatial and temporal gradients expected here the flow may be assumed irrotational. If we assume that a velocity potential ϕ exists, the inviscid equations of motion reduce to

$$\phi_{tt} + 2\phi_x\phi_{xt} + 2\phi_y\phi_{yt} + 2\phi_x\phi_y\phi_{xy} = (a^2 - \phi_x^2)\,\phi_{xx} + (a^2 - \phi_y^2)\,\phi_{yy}, \tag{1}$$

where a is the local speed of sound, which for a perfect gas it is given by the isentropic Bernoulli integral for unsteady flow:

$$a^2 = a_0^2 - (\gamma - 1)\,[\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2)], \tag{2}$$

where a_0 is the constant speed of sound in the undisturbed medium and γ is the ratio of specific heats. The velocity potential must satisfy the initial conditions

$$\phi(x,y,0) = \phi_0(x,y), \quad \phi_t(x,y,0) = \phi_1(x,y), \tag{3}$$

where the functions ϕ_0 and ϕ_1 are taken to be zero ahead of the wave front, i.e. for $x > f(y)$; immediately following the shock, their values must be consistent with the appropriate shock jump relations.

As mentioned in previous sections, we consider the shocks to be not only initially weak but also nearly straight. We take the maximum strength of the wave front to be at $y = 0$ and the pressure coefficient, $C_p(x,y,t) = -2\phi_t/a_0^2$ to lowest order, to be small for all x and y at $t = 0$. We define

$$\epsilon \equiv C_p(0^-, 0, 0) \ll 1$$

as the small parameter associated with the shock strength. For the shock to be practically straight we require that f be such that the maximum slope f' is small. If we define L to be the point of maximum slope, i.e. $f''(L) \equiv 0$, then the requirement $\delta \equiv L/R_0 \ll 1$, where R_0 is the minimum radius of curvature of the shock, ensures that the slope is small everywhere. Another restriction we shall need to place on the shape of the wave front concerns the rate of change of the radius of curvature of the wave front at $t = 0$. Denoting the radius of curvature by $R(y_i)$, we have

$$R(y_i) = [1 + f'(y_i)^2]^{\frac{3}{2}}/f''(y_i). \tag{4}$$

In §4 we shall need to require that $R_0'' R_0 \delta^2 \equiv (3 - f^{\mathrm{iv}}(0)\,R_0^3)\,\delta^2$ be of order one in the limit of vanishing δ. Here R_0'' denotes the second derivative of $R(y_i)$ at $y_i = 0$ and $f^{\mathrm{iv}}(0)$ is the fourth derivative of f at $y_i = 0$. Examples of possible wave-front shapes are

$$f_1(y;\, A, l) = A[1 - \exp(-y^2/l^2)], \quad f_2(y;\, c_1, c_2) = \tfrac{1}{2}c_1 y^2 - \tfrac{1}{30}c_2 y^6.$$

Both shapes are smooth and possess inflexion points. The parameter δ is found to be $2^{\frac{1}{2}}A/l$ in the first case and $c_1^{\frac{5}{4}}c_2^{-\frac{1}{4}}$ in the second, and it is clear that the limit $\delta \to 0$ may always be taken. The above formulae may be used to calculate $R_0''R_0\delta^2$. In the first case this equals $3(1+\delta^2)$ exactly and the requirement that $R_0''R_0\delta^2 = O(1)$ as $\delta \to 0$ is satisfied; thus the results derived in this paper may be applied to the first example. The second case is given because it is a wave-front shape which does not satisfy this condition; essentially, this is because $f_2^{\text{iv}}(0) \equiv 0$ and, consequently, $R_0''R_0\delta^2 \to 0$ as $\delta \to 0$. In the following sections we shall always assume

$$\kappa \equiv -f^{\text{iv}}(0)\,R_0^3\delta^2 \approx R_0''R_0\delta^2 = O(1).$$

Our main objective is the determination of solutions to (1) and (3) which are valid for small ϵ and δ and, for consistency, whose pressure and velocity perturbations are always small. In particular we wish to study the case where wave-front crossing occurs. In this paper we give approximations to (1) in two distinct regions. The first is just the linear wave equation and is valid for times of order L/a_0. The solution, subject to the initial conditions (3), is easily found; this is the outer solution. An approximation to (1) which is valid as the wave front approaches the arête, i.e. the cusp in the caustic, is derived in § 5. The initial condition for this equation is obtained by the method of matched asymptotic expansions, thus establishing the initial-value problem governing the flow in the vicinity of the arête.

In the next section the outer solution and the expression for the caustic shape near the cusp are derived. In § 5 the inner region in which the nonlinear effects predominate is discussed and the inner equation deduced. In § 6 the appropriate initial condition for this equation is obtained through the method of matched asymptotic expansions. These results provide a similitude which shows that the solution to the inner problem involves only a single parameter.

4. Outer region

To determine the outer equation it is convenient to work in a co-ordinate system that moves with the wave front. Accordingly, we write the full potential equation (1) in terms of the co-ordinates $X \equiv x - a_0 t$, y and t:

$$\begin{aligned}\phi_{tt} - 2a_0\phi_{Xt} + 2\phi_X\phi_{Xt} + (\gamma-1)\,\phi_t\phi_{XX} - (\gamma+1)\,a_0\phi_X\phi_{XX}& \\ = a_0^2\phi_{yy} - [2\phi_y\phi_{ty} + (\gamma-1)\,\phi_t\phi_{yy}] + a_0[2\phi_y\phi_{Xy} + (\gamma-1)\,\phi_X\phi_{yy}] + \dots,& \quad (5)\end{aligned}$$

where (2) has been used and the terms omitted are cubic in ϕ.

In the outer region it is natural to take

$$a_0 t = O(L), \quad y = O(L), \quad X = O(\Lambda^*), \quad \phi = O(a_0 k^*),$$

where the length scales Λ^* and k^* are yet to be determined. According to the theory of geometrical acoustics the square of the pressure coefficient immediately following the shock varies inversely with the ray-tube area. For our problem all the rays are straight lines and, consequently, the pressure coefficient behind the shock obeys (see, for example, Friedlander 1958, pp. 51–56)

$$C_{ps}(t;\,y_i) = C_{ps}(0;\,y_i)\left(\frac{R(y_i)}{R(y_i)-a_0t}\right)^{\frac{1}{2}}.$$

Here $R(y_i)$ is the radius of curvature of the initial wave front at the point $y = y_i$; y_i effectively labels the ray of interest. Thus, for times of order L/a_0, we see that the time rate of change of the shock's strength is of order $a_0/R(y_i)$. This suggests that the appropriate outer scaling for the derivative $(\partial/\partial t)_X$ is a_0/R_0. Accordingly, we adopt the following outer scaling for the derivatives in (5):

$$\frac{\partial}{\partial t}=\frac{a_0}{R_0}\frac{\partial}{\partial \tilde{t}},\quad \frac{\partial}{\partial y}=\frac{1}{L}\frac{\partial}{\partial \tilde{y}},\quad \frac{\partial}{\partial X}=\frac{1}{\Lambda^*}\frac{\partial}{\partial \tilde{X}}$$

and $\phi = a_0 k^* \tilde{\phi}$. For $t = O(L/a_0)$, there is no amplification to lowest order in δ; we therefore assume that k^*/Λ^* and k^*/L are small. When the above expressions are substituted in (5) we find that it may be written in the following form:

$$\tilde{\phi}_{\tilde{t}\tilde{t}}-2\frac{R_0}{\Lambda^*}\tilde{\phi}_{\tilde{X}\tilde{t}}+\frac{k^*}{\Lambda^*}\frac{R_0}{\Lambda^*}[2\tilde{\phi}_{\tilde{X}}\tilde{\phi}_{\tilde{X}\tilde{t}}+(\gamma-1)\tilde{\phi}_{\tilde{t}}\tilde{\phi}_{\tilde{X}\tilde{X}}]-(\gamma+1)\frac{k^*}{\Lambda^*}\frac{R_0^2}{\Lambda^{*2}}\tilde{\phi}_{\tilde{X}}\tilde{\phi}_{\tilde{X}\tilde{X}}$$
$$=\frac{1}{\delta^2}\tilde{\phi}_{\tilde{y}\tilde{y}}+\frac{k^*}{\Lambda^*}\frac{1}{\delta^2}[2\tilde{\phi}_{\tilde{y}}\tilde{\phi}_{\tilde{X}\tilde{y}}+(\gamma-1)\tilde{\phi}_{\tilde{X}}\tilde{\phi}_{\tilde{y}\tilde{y}}]-\frac{k^*}{\Lambda^*}\frac{1}{\delta^2}\frac{\Lambda^*}{R_0}[2\tilde{\phi}_{\tilde{y}}\tilde{\phi}_{\tilde{y}\tilde{t}}+(\gamma-1)\tilde{\phi}_{\tilde{t}}\tilde{\phi}_{\tilde{y}\tilde{y}}]+\dots.$$

We now assume that k^*/Λ^* and k^*/L are small; it may be shown that this implies that the omitted cubic terms are negligible compared with the largest of the first- and second-order terms. Furthermore, when k^*/Λ^* is small, the third term on the left-hand side and the second term on the right-hand side are seen to be negligible compared with the $\tilde{\phi}_{\tilde{X}\tilde{t}}$ term and the $\tilde{\phi}_{\tilde{y}\tilde{y}}$ term, respectively. When these terms are dropped, the terms remaining are

$$\tilde{\phi}_{\tilde{t}\tilde{t}}-2\frac{R_0}{\Lambda^*}\tilde{\phi}_{\tilde{X}\tilde{t}}-(\gamma+1)\frac{k^*}{\Lambda^*}\frac{R_0^2}{\Lambda^{*2}}\tilde{\phi}_{\tilde{X}}\tilde{\phi}_{\tilde{X}\tilde{X}}=\frac{1}{\delta^2}\tilde{\phi}_{\tilde{y}\tilde{y}}-\frac{k^*}{\Lambda^*}\frac{1}{\delta^2}\frac{\Lambda^*}{R_0}[2\tilde{\phi}_{\tilde{y}}\tilde{\phi}_{\tilde{y}\tilde{t}}+(\gamma-1)\tilde{\phi}_{\tilde{t}}\tilde{\phi}_{\tilde{y}\tilde{y}}].$$

This expression is reduced further by assuming that δ is small. An immediate consequence of this assumption is that the $\tilde{\phi}_{\tilde{t}\tilde{t}}$ term is negligible compared with the $\tilde{\phi}_{\tilde{y}\tilde{y}}$ term. Inspection of the remaining terms shows that the only choice of Λ^* which yields a non-trivial balance of terms, i.e. one which contains derivatives with respect to $\tilde{X}$, $\tilde{y}$ and $\tilde{t}$, is $\Lambda^*/R_0 = \delta^2$, or $\Lambda^* = L\delta$. Here, for convenience, we have dropped the order symbols. The resulting balance of terms may be written as

$$2\tilde{\phi}_{\tilde{X}\tilde{t}}+(\gamma+1)\frac{\epsilon}{\delta^2}\tilde{\phi}_{\tilde{X}}\tilde{\phi}_{\tilde{X}\tilde{X}}+\tilde{\phi}_{\tilde{y}\tilde{y}}=0.$$

The coefficient of the nonlinear term is obtained by noting that $\Lambda^* = \delta L$ implies that $(\partial/\partial t)_x \approx -a_0(\partial/\partial X)_t$ in the outer region and therefore that

$$C_p = O(\epsilon) = -\frac{1}{a_0^2}\phi_t \approx \frac{1}{a_0}\phi_X = O\left(\frac{k^*}{\Lambda^*}\right),$$

or simply $k^* = \Lambda^*\epsilon$. Thus, provided that $\epsilon/\delta^2 = o(1)$, nonlinear effects are negligible. We now assume that this is the case, so that the outer equation becomes

$$2\tilde{\phi}_{\tilde{X}\tilde{t}}+\tilde{\phi}_{\tilde{y}\tilde{y}}=0.$$

Transforming back to the physical variables and back to the x, y, t co-ordinates, we see that our outer equation is just the wave equation

$$\phi_{tt} = a_0^2(\phi_{xx}+\phi_{yy}), \tag{6}$$

where we have used the fact that $(\partial/\partial t)_x \approx -a_0(\partial/\partial X)_t$ in the outer region.

The explicit form of the outer solution is obtained by solving the linear wave equation (6) with the initial conditions (3). The solution to (6) which satisfies (3) is given by the well-known Poisson integral formula

$$4\pi\phi = \frac{1}{a_0^2 t} I_1 + \frac{\partial}{\partial t}\left[\frac{1}{a_0^2 t} I_0\right].$$

The quantities I_0 and I_1 are the integrals of ϕ_0 and ϕ_1 over a sphere of radius $a_0 t$ centred at the point $(x, y, 0)$. This sphere has the equation

$$(x_s - x)^2 + (y_s - y)^2 + z_s^2 = a_0^2 t^2, \tag{7}$$

where the subscript s refers to points on the sphere of integration.

We now consider the distance s measured along a ray from the initial wave front to any point $(x, y, 0)$; then

$$s = [1 + f'(y_i)^2]^{\frac{1}{2}}[x - f(y_i)] = -\frac{[1 + f'(y_i)^2]^{\frac{1}{2}}}{f'(y_i)}(y - y_i). \tag{8}$$

Here we have defined s to be positive ahead of the initial wave front and negative behind it. In the following sections it will be useful to have the functions ϕ_0 and ϕ_1 in terms of s instead of x and y. Using the fact that f/L and f' are small, we may replace (8) by

$$y_s = y_i + o(s_s), \quad s_s = x_s - f(y_s) + o(s_s). \tag{9}$$

Using (9) to replace x_s in the integrands of I_0 and I_1, we may write these integrals to lowest order as

$$I_0 = \iint \phi_0(s_s, y_s)\, dA, \quad I_1 = \iint \phi_1(s_s, y_s)\, dA;$$

the area element dA of the sphere (7) may be written as

$$dA = \frac{a_0 t\, dz_s\, dy_s}{[a_0^2 t^2 - (y_s - y)^2 - z_s^2]^{\frac{1}{2}}}.$$

Using (7) and (9), we now write s_s explicitly in terms of y_s and z_s:

$$s_s = X + a_0 t \pm [a_0^2 t^2 - (y_s - y)^2 - z_s^2]^{\frac{1}{2}} - f(y_s) + o(s_s),$$

where the $\pm$ sign refers to points on the sphere of integration with $x_s \gtrless x$, respectively.

The above results have been derived in terms of the physical variables X, y, t, etc. We formally define the outer variables

$$\tilde{t} \equiv \frac{a_0 t}{L}, \quad \tilde{y} \equiv \frac{y}{L}, \quad \tilde{z} \equiv \frac{z}{\delta^{\frac{1}{2}} L}, \quad \tilde{X} \equiv \frac{X}{\Lambda^*} = \frac{X}{\delta L}, \quad \tilde{\phi} \equiv \frac{\phi}{k^* a_0} = \frac{\phi}{\epsilon \delta L a_0}.$$

The outer solution may now be written as

$$4\pi\tilde{\phi} = \frac{1}{\tilde{t}}\tilde{I}_1 - \frac{1}{\tilde{t}}\frac{\partial}{\partial \tilde{X}}\tilde{I}_0, \tag{10}$$

where
$$\tilde{I}_0 \equiv \iint \tilde{\phi}_0(\tilde{s}_s, \tilde{y}_s)\, d\tilde{A}, \quad \tilde{I}_1 \equiv \iint \tilde{\phi}_1(\tilde{s}_s, \tilde{y}_s)\, d\tilde{A},$$

in which
$$\tilde{\phi}_0(\tilde{s}_s, \tilde{y}_s) \equiv \phi_0(s_s, y_s)/\epsilon a_0 \delta L, \quad \tilde{\phi}_1(\tilde{s}_s, \tilde{y}_s) \equiv \phi_1(s_s, y_s)/a_0^2 \epsilon$$

and
$$\tilde{s}_s(\tilde{y}_s, \tilde{z}_s; \tilde{X}, \tilde{y}, \tilde{t}) \equiv \frac{s_s}{\delta L} = \tilde{X} + \frac{1}{\delta}\tilde{t} \pm \frac{1}{\delta}[\tilde{t}^2 - (\tilde{y}_s - \tilde{y})^2 - \delta \tilde{z}_s^2]^{\frac{1}{2}} - \tilde{f}(\tilde{y}_s), \tag{11}$$

$$d\tilde{A} \equiv dA/\delta L^2.$$

In determining the scaling for s_s, we have used the fact that the functions ϕ_0 and ϕ_1 are identically zero for $s_s > 0$; therefore it is necessary to consider only non-positive values of s_s in the above integrations. As a result of this, s_s is of order δL in the outer region and, although $\tilde{y}_s$ and $\tilde{y}$ are each of order one in the outer region, their difference $y_s - y$ will always be of order $\delta^{\frac{1}{2}}L$ there. Because z_s is also found to be of order $\delta^{\frac{1}{2}}L$, the last two terms under the square root are of the same order. In the outer region $\tilde{t}$ can vanish; (11) is therefore the lowest-order expression for $\tilde{s}_s$ which is uniformly valid in the outer region.

Strictly speaking, the outer solution is valid only for times of order L/a_0. The result (10), of course, predicts infinite pressures when the wave front reaches the caustic surface. This surface is defined by the intersection of adjacent rays, i.e. of adjacent normals to the initial wave front. These considerations imply that the distance, measured along a ray, from a point on the initial wave front to the caustic is just $R(y_i)$, the local radius of curvature of the wave front. In terms of the Cartesian co-ordinates x and y the equation of the caustic surfaces is found by substituting $s = R(y_i)$ in (8). Doing so and expanding for small δ, we find that the caustic is cusped and that near this cusp it has the shape

$$\frac{x_c - R_0}{R_0} = \tfrac{1}{2}R_0'' R_0 \delta^2 \left(\frac{y_i}{L}\right)^2, \quad \frac{y_c}{L} = -\tfrac{1}{3}R_0'' R_0 \delta^2 \left(\frac{y_i}{L}\right)^3$$

to lowest order. Or, if we eliminate y_i,

$$\left(\frac{x_c - R_0}{R_0}\right)^3 = \tfrac{9}{8}R_0'' R_0 \delta^2 \left(\frac{y_c}{L}\right)^2,$$

where the subscript c refers to the caustic. As discussed in § 3, we require that $\kappa \approx R_0'' R_0 \delta^2$ be of order one. Initial shock shapes which have $\kappa = o(1)$, such as the second example discussed in § 3, require the inclusion of higher-order terms in the above expansions. This will change the relative sizes of the x and y in the inner scaling, thereby increasing the strength of the singularity at the focal point. For wave fronts with $\kappa = o(1)$ but such that a δ may be found and made small, the procedure of this paper may be used to obtain analogous results.

The singular behaviour of the solution at the caustic suggests that nonlinear effects are important there. In the next section we assume that the initial strength of the wave front is so small that these nonlinear effects are important only in the vicinity of the caustic. We then find the appropriate nonlinear equation governing the flow near the cusp in the caustic.

5. Inner region

We now seek an inner expansion valid in the vicinity of the arête. We introduce the inner variables $\hat{x}, \hat{t}, \hat{y}$, χ and $\hat{\phi}$ and corresponding inner scales Δ, λ, Λ and k, viz.

$$\hat{x} \equiv \frac{x - R_0}{\Delta R_0}, \quad \hat{t} \equiv \frac{a_0 t - R_0}{\Delta R_0}, \quad \hat{y} \equiv \frac{y}{\lambda}, \quad \chi \equiv \frac{X}{\Lambda}, \quad \hat{\phi} \equiv \frac{\phi}{a_0 k}. \tag{12}$$

We assume that the nonlinear effects are important in only a small neighbourhood of the cusp, and we shall therefore take $\Delta = o(1)$ and $\lambda/L = o(1)$. The amplification is greatest in the region immediately following the shock and we expect nonlinear effects to be most important there; consequently, we use the same scaling for x and t. Furthermore, we assume that, although the strength of the shock is considerably amplified in the focal region, it remains small. Thus we shall take $\phi_X/a_0 = O(k/\Lambda) = o(1)$ and $\phi_y/a_0 = O(k/\lambda) = o(1)$. When the scaling (12) is substituted into (5) the full potential equation may be written as

$$\hat{\phi}_{\hat{t}\hat{t}} - 2\frac{\Delta R_0}{\Lambda}\hat{\phi}_{\chi\hat{t}} + \frac{k}{\Lambda}\frac{\Delta R_0}{\Lambda}[2\hat{\phi}_\chi\hat{\phi}_{\chi\hat{t}} + (\gamma-1)\hat{\phi}_{\hat{t}}\hat{\phi}_{\chi\chi}] - (\gamma+1)\frac{k}{\Lambda}\frac{\Delta^2R_0^2}{\Lambda^2}\hat{\phi}_\chi\hat{\phi}_{\chi\chi}$$

$$= \frac{\Delta^2R_0^2}{\lambda^2}\hat{\phi}_{\hat{y}\hat{y}} + \frac{k}{\Lambda}\frac{\Delta^2R_0^2}{\lambda^2}[2\hat{\phi}_{\hat{y}}\hat{\phi}_{\chi\hat{y}} + (\gamma-1)\hat{\phi}_\chi\hat{\phi}_{\hat{y}\hat{y}}]$$

$$-\frac{k}{\Lambda}\frac{\Delta^2R_0^2}{\lambda^2}\frac{\Lambda}{\Delta R_0}[2\hat{\phi}_{\hat{y}}\hat{\phi}_{\hat{y}\hat{t}} + (\gamma-1)\hat{\phi}_{\hat{t}}\hat{\phi}_{\hat{y}\hat{y}}] + \dots.$$

As in our analysis of the outer region, the assumptions that k/Λ and k/λ are $o(1)$ imply that we may drop not only the cubic terms, but also the third term on the left-hand side and the second term on the right-hand side. The resulting equation reads

$$\hat{\phi}_{\hat{t}\hat{t}} - 2\frac{\Delta R_0}{\Lambda}\hat{\phi}_{\chi\hat{t}} - (\gamma+1)\frac{k}{\Lambda}\frac{\Delta^2R_0^2}{\Lambda^2}\hat{\phi}_\chi\hat{\phi}_{\chi\chi}$$

$$= \frac{\Delta^2R_0^2}{\lambda^2}\hat{\phi}_{\hat{y}\hat{y}} - \frac{k}{\Lambda}\frac{\Delta^2R_0^2}{\lambda^2}\frac{\Lambda}{\Delta R_0}[2\hat{\phi}_{\hat{y}}\hat{\phi}_{\hat{y}\hat{t}} + (\gamma-1)\hat{\phi}_{\hat{t}}\hat{\phi}_{\hat{y}\hat{y}}].$$

On physical grounds it is clear that we must require that the inner equation contains χ, $\hat{y}$ and $\hat{t}$ derivatives and at least one nonlinear term; the only choice of Λ which results in such an equation satisfies $\Lambda/\Delta R_0 = o(1)$. For small $\Lambda/\Delta R_0$, the nonlinear term on the right-hand side and the $\hat{\phi}_{\hat{t}\hat{t}}$ term may be dropped. In order to balance the remaining terms we need

$$\frac{\Delta R_0}{\Lambda} = \frac{\Delta^2R_0^2}{\lambda^2} = \frac{k}{\Lambda}\frac{\Delta^2R_0^2}{\Lambda^2} \gg 1. \tag{13}$$

Thus, to lowest order, the inner equation is

$$2\hat{\phi}_{\chi\hat{t}} + (\gamma+1)\hat{\phi}_\chi\hat{\phi}_{\chi\chi} + \hat{\phi}_{\hat{y}\hat{y}} = 0. \tag{14}$$

We remark here that (14) also describes low frequency, unsteady, transonic flows, as one might have anticipated.

We now assume that in the inner region the relative size of the x and y length scales is given by the caustic surface calculated in §4. This requires that we take $\Delta^3 = O(\lambda^2/L^2)$. Dropping the order symbol and simply substituting $\lambda^2 = \Delta^3L^2$ in (13), we may express λ, Λ and k in terms of Δ and δ:

$$\lambda/L = \Delta^{\frac{3}{2}}, \quad \Lambda/L = \Delta^2\delta, \quad k/\Lambda = \Delta\delta^2. \tag{15}$$

Thus Λ, λ and k are known in terms of the physical parameters ϵ and δ once Δ has been determined.

The flow in the focal region is found by solving the inner equation (14) subject to an appropriate initial condition. In the next section we determine this initial condition through matching with the outer solution.

6. The matching and the similitude

In this section the initial data for the inner problem are obtained through the method of matched asymptotic expansions. It is further shown that, apart from a simple scaling of the independent and dependent variables, the solution to the resultant initial-value problem depends on only a single combination of the physical parameters: the similarity parameter.

We first consider the outer expansion, which we shall write in the inner variables (12) and then expand for small Δ and δ with the inner variables fixed. We begin by expressing (11) in terms of the inner variables (12) and expanding for small Δ and δ. As a result of this expansion we find that the inner scaling for the integration variables y_s and z_s is given by

$$\hat{y}_s = \Delta^{-\frac{1}{2}}\tilde{y}_s = \Delta^{-\frac{1}{2}}L^{-1}y_s,$$
$$\hat{z}_s = \delta^{\frac{1}{2}}\Delta^{-1}\tilde{z}_s = \Delta^{-1}L^{-1}z_s$$

and that, to lowest order, the inner expansion of $\tilde{s}_s$ is

$$\tilde{s}_s(\Delta^{\frac{1}{2}}\hat{y}_s,\, \Delta\delta^{-\frac{1}{2}}\hat{z}_s;\, \Delta^2\chi,\, \Delta^{\frac{1}{2}}\hat{y}_s,\, (1+\Delta\hat{t})/\delta) = \Delta^2[\tfrac{1}{2}\hat{z}_s^2 - \beta(\hat{y}_s)], \tag{16}$$

where $\beta(\hat{y}_s)$ is defined by

$$\beta(\hat{y}_s) = \beta(\hat{y}_s;\, \chi, \hat{y}, \hat{t}) \equiv -\tfrac{1}{24}\kappa\hat{y}_s^4 + \tfrac{1}{2}\hat{t}\hat{y}_s^2 + \hat{y}_s\hat{y} - \chi.$$

In the derivation of (16) we have chosen the lower sign in (11). Any other choice corresponds to portions of the integration sphere with $x_s > f(y_s)$ and contributes nothing to the integrals $\tilde{I}_0$ and $\tilde{I}_1$.

The integrals $\tilde{I}_0$ and $\tilde{I}_1$ are to be evaluated over the surface of a sphere. However, when the inner expansion of the outer area element $d\tilde{A}$ is taken, we see that it may be replaced by $\Delta^{\frac{3}{2}}\delta^{-1}\,d\hat{z}_s d\hat{y}_s$. Thus, in the inner expansions of $\tilde{I}_0$ and $\tilde{I}_1$, we may transfer the integration from the sphere to the $\hat{z}_s$, $\hat{y}_s$ plane.

We need the expansion of $\tilde{\phi}$ near the cusp only for times less than R_0/a_0, i.e. $\hat{t} < 0$. The area of integration will therefore be bounded by the single closed curve $\tilde{s}_s = 0$, or $\hat{z}_s = \pm[2\beta(\hat{y}_s)]^{\frac{1}{2}}$, as sketched in figure 2. The $\hat{z}_s = 0$ intercepts $\hat{y}_u$ and $\hat{y}_l$ are just the two real roots of $\beta = 0$.

When the inner expansions of the integrals $\tilde{I}_0$ and $\tilde{I}_1$ are taken it will be useful to have the Taylor series expansions of $\tilde{\phi}_0$ and $\tilde{\phi}_1$ for small $\tilde{s}_s$ and $\tilde{y}_s$:

$$\tilde{\phi}_0 = \tfrac{1}{2}\tilde{s}_s[1 + O(\tilde{s}_s, \tilde{y}_s^2)], \quad \tilde{\phi}_1 = -\tfrac{1}{2}[1 + O(\tilde{s}_s, \tilde{y}_s^2)].$$

These are, of course, consistent with the weak-shock jump conditions.

The above results may now be used to show that the inner expansions of $\tilde{I}_0$ and $\tilde{I}_1$ are

$$\tilde{I}_0 = \tfrac{1}{2}\Delta^{\frac{7}{2}}\delta^{-1}\int_{\hat{y}_l}^{\hat{y}_u}\int_0^{(2\beta)^{\frac{1}{2}}}[\hat{z}_s^2 - 2\beta(\hat{y}_s)]\,d\hat{z}_s d\hat{y}_s$$

and

$$\tilde{I}_1 = -\Delta^{\frac{3}{2}}\delta^{-1}\int_{\hat{y}_l}^{\hat{y}_u}\int_0^{(2\beta)^{\frac{1}{2}}} d\hat{z}_s d\hat{y}_s = -\Delta^{-2}\frac{\partial \tilde{I}_0}{\partial \chi}$$

to lowest order. When the $\hat{z}_s$ integration is performed and the result substituted in (10), we find

$$\tilde{\phi}\left(\Delta^2\chi, \Delta^{\frac{1}{2}}\hat{y}, \frac{1+\Delta\hat{t}}{\delta}\right) = -\frac{1}{\pi \times 2^{\frac{1}{2}}}\Delta^{\frac{3}{2}}\int_{\hat{y}_l}^{\hat{y}_u}\beta^{\frac{1}{2}}(\hat{y}_s)\,d\hat{y}_s.$$

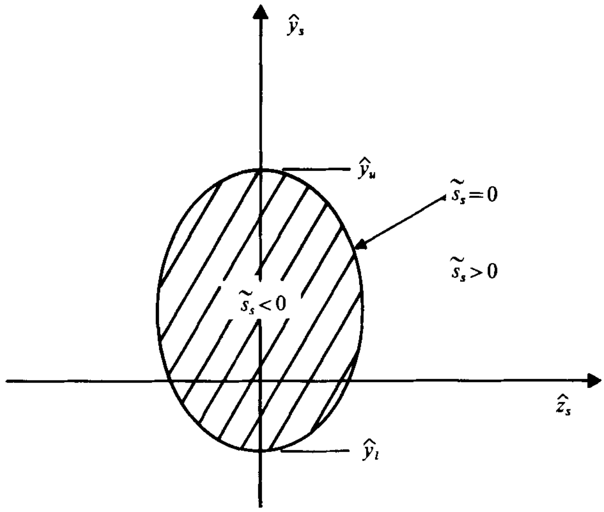


FIGURE 2. Area of integration for I_0 and I_1.

Thus, when written in inner variables, the inner expansion of the outer solution is given by

$$\tilde{\phi} \equiv \frac{k^*}{k}\tilde{\phi}\left(\Delta^2\chi, \Delta^{\frac{3}{2}}\hat{y}, \frac{1+\Delta\hat{t}}{\delta}\right) = -\frac{1}{\pi\times 2^{\frac{1}{2}}}\frac{\epsilon}{\Delta^{\frac{3}{2}}\delta^2}\int_{\hat{y}_l}^{\hat{y}_u}\beta^{\frac{1}{2}}\,d\hat{y}_s. \tag{17}$$

This result is essentially the expansion of the linear solution (10) as the wave front approaches the cusp in the caustic. As mentioned in previous sections, the pressure distribution associated with this pressure field can be singular. The pressure coefficient based on (17) is

$$C_p = \frac{1}{\pi\times 2^{\frac{1}{2}}}\frac{\epsilon}{\Delta^{\frac{1}{2}}}\int_{\hat{y}_l}^{\hat{y}_u}\beta^{-\frac{1}{2}}\,d\hat{y}_s.$$

At $t = R_0/a_0$ or, in terms of the inner variables, $\hat{t} = 0$, the pressure coefficient is proportional to $(R_0-x)^{-\frac{1}{4}}$ along the x axis and to $|y|^{-\frac{1}{3}}$ along the y axis. This is more singular than the analogous result for a smooth caustic, where the pressure behaves as the $-\frac{1}{6}$ power of the distance along the caustic and the $-\frac{1}{4}$ power of the distance normal to it (see, for example, Friedlander 1958, pp. 67–70; or Hayes 1968).

Another interesting feature of (17), which turns out to be essential for the matching, is that it is self-similar in time, i.e. it may be written as

$$\hat{\tilde{\phi}} = (-\hat{t})^{\frac{3}{2}}F(\chi/\hat{t}^2, \hat{y}/(-\hat{t})^{\frac{3}{2}}).$$

To show this, we define similarity variables σ and Γ by

$$\sigma \equiv -\tfrac{2}{3}\xi/\tau^2, \quad \Gamma \equiv 24\eta/(-6\tau)^{\frac{3}{2}},$$

where $\xi \equiv \chi/\kappa$, $\eta \equiv \hat{y}/\kappa$ and $\tau \equiv \hat{t}/\kappa$. Using these definitions and replacing the integration variable $\hat{y}_s$ in (17) by $(-6\tau)^{\frac{1}{2}}q$, we find that (17) may be rewritten as

$$\hat{\tilde{\phi}} = -\tfrac{1}{2}(\tfrac{1}{2}\kappa)^{\frac{1}{2}}\frac{\epsilon}{\Delta^{\frac{3}{2}}\delta^2}(-\tau)^{\frac{3}{2}}G(\sigma,\Gamma), \tag{18}$$

where G is defined by

$$G(\sigma,\Gamma) \equiv \frac{6}{\pi}\int_{q_l}^{q_u}\bar{\beta}^{\frac{1}{2}}(q)\,dq,$$

in which $\bar{\beta} \equiv -q^4 - 2q^2 + \Gamma q + \sigma$ and q_u and q_l are the two real roots of $\bar{\beta} = 0$.

We now consider the inner solution. Although, in general, solutions of the form (18) do not satisfy (14), the inner solution must have such a form as $\hat{t} \to -\infty$. Thus it is most convenient to do the matching in the co-ordinates σ, Γ and τ. Accordingly, we write the solution $\hat{\phi}(\chi, \hat{y}, \hat{t})$ to (14) in terms of σ, Γ and τ and then rewrite $\hat{\phi}$ in outer variables:

$$\hat{\phi} = \hat{\phi}(\sigma, \Gamma, \tau) = \hat{\phi}(\tilde{\sigma}, \tilde{\Gamma}, (\delta\tilde{t} - 1)/\Delta\kappa),$$

where $\tilde{\sigma} \equiv -\frac{2}{3}\tilde{X}/(\delta\tilde{t} - 1)^2$ and $\tilde{\Gamma} \equiv 4\tilde{y}/6^{\frac{1}{2}}(1 - \delta\tilde{t})^{\frac{3}{2}}$. The usual matching principle states that, in the limit $\epsilon \to 0$, $\Delta \to 0$, $\delta \to 0$ with $\tilde{X}$, $\tilde{y}$ and $\tilde{t}$ fixed, $\hat{\phi}(\tilde{\sigma}, \tilde{\Gamma}, (\delta\tilde{t} - 1)/\Delta\kappa)$ must approach (18) asymptotically. Thus the function $\hat{\phi}(\sigma, \Gamma, \tau)$ must satisfy

$$\hat{\phi}(\sigma, \Gamma, \tau) \sim -\tfrac{1}{2}(\tfrac{1}{2}\kappa)^{\frac{1}{2}}(-\tau)^{\frac{3}{2}}\, G(\sigma, \Gamma)$$

as $\tau \to -\infty$ for all values of σ and Γ. In order that the initial condition be non-zero and finite, we must take

$$\Delta^{\frac{3}{2}} \equiv \epsilon/\delta^2 = o(1).$$

This last result determines the scaling (15) completely in terms of the physical parameters ϵ and δ, viz.

$$\frac{\lambda}{L} = \Delta^{\frac{3}{2}} = \frac{\epsilon}{\delta^2}, \quad \frac{\Lambda}{L} = \frac{\epsilon^{\frac{1}{3}}}{\delta^{\frac{2}{3}}}, \quad \frac{k}{\Lambda} = \epsilon^{\frac{1}{3}}\delta^{\frac{2}{3}}.$$

When written in terms of the variables ξ, η and τ the initial-value problem for the inner region is

$$2\hat{\phi}_{\xi\tau} + (\gamma + 1)\kappa^{-1}\hat{\phi}_{\xi}\hat{\phi}_{\xi\xi} + \hat{\phi}_{\eta\eta} = 0,$$

where as $\tau \to -\infty$

$$\hat{\phi}(\xi, \eta, \tau) \sim -\tfrac{1}{2}(\tfrac{1}{2}\kappa)^{\frac{1}{2}}(-\tau)^{\frac{3}{2}}\, G(\xi, \eta, \tau)$$

for all values of ξ and η. Here $G(\xi, \eta, \tau)$ is just the integral $G(\sigma, \Gamma)$ rewritten in terms of the variables ξ, η and τ.

We have now established the initial-value problem governing the flow in the focal region. The solution to this problem is seen to possess a similitude, i.e. it depends on only a single combination of the remaining physical parameters γ and κ. This is readily seen when the above initial-value problem is recast in terms of the scaled velocity potential

$$\Phi \equiv \tfrac{1}{2}(\tfrac{1}{2}\kappa)^{\frac{1}{2}}\hat{\phi}.$$

The problem then becomes

$$2\Phi_{\xi\tau} + Q\Phi_{\xi}\Phi_{\xi\xi} + \Phi_{\eta\eta} = 0,$$

where as $\tau \to -\infty$

$$\Phi \sim -(-\tau)^{\frac{3}{2}}\, G(\xi, \eta, \tau).$$

The similarity parameter Q is defined by

$$Q \equiv \tfrac{1}{2}(\gamma + 1)/(2\kappa)^{\frac{1}{2}}.$$

Except for a scaling of the dependent and independent variables, any two flows with the same value of Q will be identical. The flow quantity of especial interest here is the pressure coefficient, which is given by

$$C_p = \frac{\epsilon}{(2\kappa\Delta)^{\frac{1}{2}}}\,\Phi_{\xi}(\xi, \eta, \tau;\, Q) = \frac{\epsilon^{\frac{2}{3}}\delta^{\frac{2}{3}}}{(2\kappa)^{\frac{1}{2}}}\,\Phi_{\xi}(\xi, \eta, \tau;\, Q).$$

From this we see that the pressure levels at an arête are of order $\epsilon^{\frac{2}{3}}\delta^{\frac{2}{3}}$ and the amplification is of order $\epsilon^{-\frac{1}{3}}\delta^{\frac{2}{3}}$. The result that the pressure is proportional to the two-thirds power of the initial strength of the shock was also deduced by Pierce (1971).

We conclude this section with some remarks regarding the requirement that $\epsilon = o(\delta^2)$. An important result, but one which is outside the scope of the present theory, is the prediction of the transition shock strength; that is, for a given initial shock shape, the prediction of the initial shock strength above which the straightening of the shock associated with shock dynamic theory occurs. In §4 we have seen that when

$$\epsilon = O(\delta^2) = o(1)$$

nonlinear effects are important even for times of order L/a_0. Because the distance to the caustic is large compared with L, we expect that the shock straightens without focusing. On the basis of this, we conjecture that the order of magnitude of the transition shock strength is given by $\epsilon = O(\delta^2)$. Of course, a more precise estimate must be given either by laboratory or numerical experiments or by a more comprehensive analysis.

7. Conclusion

The focusing of a very weak and almost straight shock at an arête has been examined. The method of matched asymptotic expansions has been used to establish the initial-value problem and similitude governing the flow in the focal region. The fundamental parameters in this problem are seen to be ϵ, a measure of the initial strength of the shock, and δ, which measures the rate at which the wave front converges. The maximum pressures at the arête have been shown to be proportional to $(\epsilon\delta)^{\frac{2}{3}}$. The results of this paper are valid for wave fronts with $\epsilon = o(\delta^2)$ and $\delta = o(1)$.

This research was supported by the office of Naval Research through contract N0014-76-C-0182 and the Air Force Office of Scientific Research through grant 76-2954B.

REFERENCES

BEASLEY, W. D., BROOKS, J. D. & BARGER, R. L. 1969 A laboratory investigation of N-wave focusing. *N.A.S.A. Tech. Note* D-5306.

CORNET, E. P. 1972 Focusing of an N-wave by a spherical mirror. *Appl. Res. Lab., Univ. Texas, Austin, Rep.* ARL-TR-72-40.

FRIEDLANDER, F. G. 1958 *Sound Pulses.* Cambridge University Press.

GUIRAUD, J. P. 1965 Acoustique géometrique, bruit ballistique des avions supersoniques et focalisation. *J. Méc.* **4**, 215–267.

HAYES, W. D. 1957 The vorticity jump across a gasdynamic discontinuity. *J. Fluid Mech.* **2**, 595–600.

HAYES, W. D. 1968 Similarity rules for nonlinear acoustic propagation through a caustic. *2nd Conf. Sonic Boom Res.* (ed. I. R. Schwartz). *N.A.S.A. Special Publ.* no. 180, pp. 165–171.

PIERCE, A. D. 1971. Maximum overpressures of sonic booms near the cusps of caustics. In *Noise and Vibration Control Engineering* (ed. M. J. Crocker), pp. 478–487. Purdue University Press.

STURTEVANT, B. & KULKARNY, V. A. 1974 Dynamics of weak shock waves at a focus. *Proc. 2nd Inter-Agency Symp. Univ. Res. Transportation Noise, North Carolina State Univ., Raleigh*, pp. 402–415.

STURTEVANT, B. & KULKARNY, V. A. 1976 The focusing of weak shock waves. *J. Fluid Mech.* **73**, 651–671.

WANNER, J. C., VALLEE, J., VIVIER, C. & THERY, C. 1972 Theoretical and experimental studies of the focus of sonic booms. *J. Acoust. Soc. Am.* **52**, 13–32.

WHITHAM, G. B. 1957 A new approach to the problems of shock dynamics. Part 1. Two-dimensional problems. *J. Fluid Mech.* **2**, 146–171.

WHITHAM, G. B. 1959 A new approach to the problems of shock dynamics. Part 2. Three-dimensional problems. *J. Fluid Mech.* **5**, 369–386.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves*. Interscience.